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Comparative Study of Topology Optimization Techniques

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This paper presents a comparison between three continuum-based topology optimization methods: the hybrid cellular automaton method, the optimality criteria method, and the method of moving asymptotes. The purpose of the study is to highlight the differences between the three. The optimality criteria method and the method of moving asymptotes are well established in topology optimization. The hybrid cellular automaton method is a recently developed gradient-free technique that combines both local design rules based on the cellular automaton paradigm and the finite element analysis. The closed-loop controllers used in the hybrid cellular automaton method are used to modify the mass distribution in the design domain to find an optimum material layout. The hybrid cellular automaton and optimality criteria methods and the method of moving asymptotes are described and applied in a comparative study to three sample problems. The influence of different algorithm control parameters is shown in this work. The paper demonstrates that, for the sample problems presented, the hybrid cellular automaton method generally required the fewest number of iterations to converge to a solution compared with the optimality criteria method and the method of moving asymptotes. The final topologies generated using the hybrid cellular automaton method typically had the lowest compliance and exhibited the fewest number of intermediate densities at the solution.

I. Introduction

THE field of topology optimization, or layout optimization, has been developed extensively over the past two decades and has successfully addressed a multitude of problems, from simple structural problems to vehicle crashworthiness design [1]. This class of optimization techniques can be viewed as a method for developing an initial, conceptual design by determining the best arrangement of material within a design domain. Shape optimization can be used to modify boundaries to generate optimal structures as done using level set methods [2]. As the applications of topology optimization become more complex, the need for more efficient methodologies becomes necessary. Therefore, research is now focused on developing new efficient methods for topology optimization. Topology optimization problems have been solved using several different approaches, including optimality criteria (OC) methods [3] and the method of moving asymptotes (MMA) [4], among others. These algorithms typically require gradient information to obtain the final solution. However, the analytic expression for the gradients can be readily obtained for problems that assume elastic material behavior (due to small deformations) and static loading conditions that use a linear finite element analysis (FEA). Filtering techniques typically must be applied to these gradients to reduce the occurrence of numerical issues in topology optimization.

In the application of topology synthesis to problems requiring nonlinear FEA, for example, a compliant mechanism or crashworthiness design, obtaining sensitivities can be computationally expensive. Analytic sensitivities become difficult to derive, and so assumptions are made and simplifications of the complex nonlinear interactions are used so that expressions for the sensitivities can be derived as done in Min et al. [5]. When using high-fidelity commercial finite element software packages to simulate nonlinear transient events, such as LS-DYNA [6], these sensitivities become prohibitively expensive to calculate numerically [7]. Adjoint methods and semi-analytic sensitivities can be used to efficiently compute sensitivities for complex problems if available [8–10]. An approach that requires no gradient information and uses the cellular automata (CA) paradigm is the hybrid cellular automata (HCA) method. Application of the HCA method for nonlinear transient problems has shown promise in a recent study by Patel et al. [11].

In this paper, a comparative study is performed to demonstrate the differences between the HCA method [12], the 99 line MATLAB® [13] implementation of the OC method presented by Sigmund [14], and a similar implementation of the MMA. This investigation compares the performances of the methods for the linear-static design problems.

II. Topology Optimization

The roots of topology optimization date back to the late 1980s [15,16]. Topology optimization is an iterative process that determines the best arrangement of a limited volume of structural material within a given spatial domain so as to obtain optimal mechanical performance of the design concept. The optimization process systematically redistributes material throughout the domain to minimize or maximize a specified objective. This computational technique for the optimal distribution of material in continuum structures was first introduced by Bendsøe and Kikuchi [15]. Previous work in topology optimization generally dealt with simple problems that used the assumptions of elastic material properties,

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linear deformations, and static loading conditions. Comprehensive reviews of topology optimization can be found in literature by Bendsøe and Sigmund [17], Rozvany [18], and Eschenaueuer and Olhoff [19].

The objective of the problems discussed in this paper is to achieve minimum mean compliance with a constraint on the available mass, or volume V. This can be expressed formally as

$$\begin{aligned} &\text{minimize}_{\rho_i} \; c = \mathbf{u}^T\mathbf{F} = \mathbf{u}^T\mathbf{K}\mathbf{u} \\ &\text{subject to} \; \sum_{i=1}^{N} \rho_i v_i \le V \\ &\rho_{\min} \le \rho_i \le \rho_{\max} \end{aligned} \tag{1}$$

where ρ_i is ith material element in the design domain, $\mathbf{K}$ is the global stiffness matrix, $\mathbf{u}$ is the vector of global displacements, and $\mathbf{F}$ is the vector of external global forces. The lower bound on density $\rho_{\min}$ is greater than zero to prevent singularities in the stiffness matrix as used by the FEA and to allow for the possibility of material reintroduction. Topology optimization can be performed using continuum discretizations as well as frame and truss discretizations. In this paper, the methods discussed use continuum discretizations.

Topology optimization methods fall into the categories of mathematical programming (MP), OC, and evolutionary programming methods [17]. MP techniques are mathematically based methods that are applied to problems for which the goal is to minimize or maximize an objective function. OC methods are derived from the Karush–Kuhn–Tucker (KKT) optimality conditions. Evolutionary methods are heuristic, or intuition-based, approaches that use mechanisms inspired by biological evolution, such as reproduction, mutation, and survival of the fittest, to find an optimal solution to a problem. An important distinction between the classes of methods is that MP and OC methods use continuous design variables whereas evolutionary methods use discrete design variables.

A. Material Parameterization

The ultimate goal of topology optimization is to determine a material distribution within the design domain to achieve a specified objective. For continuum structures, the homogenization and density approaches are the two most popular material parameterizations. The homogenization approach uses composite materials as the basis to describe varying material properties, wherein each element is a microstructure [15]. The density approach deals with the more direct approach of associating just one design variable with each individual material element. The material model is defined to allow the material to assume intermediate property values by defining an interpolation function. The design variables are the relative densities x_i of the elements where 0 represents a void and 1 is full density. The density of an element can be expressed as

$$\rho_i(x_i) = x_i \rho_0 \qquad (0 < x_i \le 1) \tag{2}$$

where ρ_0 is the density of the base material. In using the finite element method (FEM), the design variable is mapped to the global stiffness matrix by relating the relative density of an element to its elastic modulus.

A common method for interpolating the elastic modulus of material elements with intermediate densities ($0 < x < 1$) is the solid isotropic material with penalization (SIMP) model [20]. This is a power law that heuristically relates the relative density to the elastic modulus of each element using the following expression:

$$E_i(x_i) = x_i^p E_0 \qquad (p \ge 1) \tag{3}$$

where p is the penalization parameter and E_0 is the elastic modulus of the base isotropic material. Therefore, the elements of differing relative densities in the design domain can be viewed as unique isotropic material elements. The power p is used to penalize intermediate densities to drive the elemental densities within the design domain to either have full density ($x = 1$) or have no density ($x = 0$). Most optimizers require this penalization to generate 0–1 topologies (note: the value of the relative density lower bound $x_{\min}$ used in this paper is 0.001). Although the density approach is used in gradient-based optimization methods because the material mapping it is treated as a continuous function, this material parametrization can be used with nongradient methods so that the material is distributed in a continuous manner from one iteration to the next. This allows the topology to evolve in a smooth manner.

B. Multiple Loading Conditions

When a design problem is posed in which loading exists in multiple, independent scenarios, an analysis must be performed for each load case. In conventional minimum compliance topology optimization problems, a weighted sum of the compliance from each load case for each element is used to represent the final compliance [17]. Thus, for all method presented in this study, the final compliance of an element in the design domain is represented by

$$\hat{c}_i = \sum_{L=1}^{N_L} \alpha_L c_i^L \tag{4}$$

where c_i^L is the compliance of the ith element for load case L and N_L is the total number of load cases. The resulting field state of a cell is the resulting aggregate compliance, that is, $S_i \equiv \hat{c}_i$. The load cases are weighted using the weight parameter α_L. The influence of a load case can be varied by modifying this parameter.

C. Numerical Issues

Three primary numerical issues exist in topology optimization: checkerboards, mesh dependency, and local minima. The checkerboarding phenomenon refers to the alternating full density and void material elements. This results from the optimizer taking advantage of the artificially high stiffness that this layout represents in the FEA. The easiest way to prevent checkerboarding is to use higher-order elements (8- or 9-node elements) if the penalization is small enough, although this increases computational time [21,22]. Mesh dependency refers to the problem of generating different topologies for different discretizations of the design domain. Borrowing from image processing, filtering techniques are most commonly employed to alleviate both checkerboarding and the mesh dependence problem [22].

III. Optimality Criteria Methods

An OC method is derived from the KKT optimality conditions. The origin of OC methods in topology optimization dates back to the pioneering work of Bendsøe and Kikuchi [15]. An OC method is based on a heuristic fixed-point-type updating scheme and is primarily suited for problems with a small number of constraints compared with the number of design variables. In general, these methods are more computationally efficient than conventional MP methods [20]. Because the material volume constraint is the only active constraint, an OC method can be used to provide a more rapid convergence compared with other optimization schemes.

To illustrate the attributes of the OC method presented by Bendsøe, we will follow the derivation that is presented in Ananiev [23]. Discretizing the problem with finite elements, similar to the formulation stated in Eq. (1), the optimization problem can be formulated as

$$\begin{aligned} &\min_x \; \mathbf{u}^T\mathbf{K}\mathbf{u} \\ &\text{s.t.} \; \sum_{i=1}^{N} x_i v_i = V \\ &\mathbf{K}\mathbf{u} = \mathbf{F} \\ &0 < x_i \le 1 \end{aligned} \tag{5}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{u}$ is the vector of global displacements, and $\mathbf{F}$ is the vector of external global forces. The design variables x_i are defined by Eq. (2) and N is the number of material elements in the design domain. The formulation includes a volume constraint, and each element has unit volume v_i. The following Lagrangian function with respect to densities, displacements, and multipliers can be written as

$$\mathcal{L} = \mathbf{u}^T\mathbf{K}\mathbf{u} + \lambda\left(\sum_{i=1}^{N} x_i v_i - V\right) + \mu(\mathbf{K}\mathbf{u} - \mathbf{F}) + \sum_{i=1}^{N} \epsilon_i(-x_i + 0) + \sum_{i=1}^{N} \kappa_i(x_i - 1) \tag{6}$$

where λ, μ, ϵ_i, and κ_i are the Lagrangian multipliers corresponding to the given constraints. The KKT optimality conditions for this problem are found at the stationary points of this function. The optimality condition with respect to x_i is as follows:

$$\begin{gathered} \frac{\partial L}{\partial x_i} = -\mathbf{u}_i^T \frac{\partial \mathbf{K}_i}{\partial x_i}\mathbf{u}_i + \lambda - \epsilon_i + \kappa_i = 0 \\ \sum_{i=1}^{N} x_i v_i - V = 0 \\ -x_i + 0 = 0 \quad \text{or} \quad x_i - 1 = 0 \end{gathered} \tag{7}$$

When a bound constraint is not active, the corresponding multiplier (ϵ or κ) is equal to zero and the optimality condition simplifies to

$$B_i \equiv \frac{1}{\lambda}\mathbf{u}_i^T \frac{\partial \mathbf{K_I}}{\partial x_i}\mathbf{u} = \frac{1}{\lambda}\frac{\partial c}{\partial x_i} = 1 \tag{8}$$

Finally, without further explanation in the literature, an update that uses a move limit and is based on the aforementioned condition results. The density update at the $k+1$ iteration for the ith element is expressed as

$$x_i^{(k+1)} = \begin{cases} \max\{(1-\zeta)x_i^{(k)}, x_{\min}\} & \text{if } (x_i^{(k)} B_i^{(k)^\eta} < \max\{(1-\zeta)x_i^{(k)}, x_{\min}\}) \\ \min\{(1+\zeta)x_i^{(k)}, x_{\max}\} & \text{if } (x_i^{(k)} B_i^{(k)^\eta} > \min\{(1+\zeta)x_i^{(k)}, x_{\max}\}) \\ x_i^{(k)} B_i^{(k)^\eta} & \text{otherwise} \end{cases} \tag{9}$$

where

$$B_i^{(k)} \equiv \frac{1}{\lambda^{(k)}}\mathbf{u}_i^T \frac{\partial \mathbf{K}_i}{\partial x_i}\mathbf{u} \tag{10}$$

The value of the relative density upper bound $x_{\max}$ is 1. The Lagrangian multiplier λ is calculated at each iteration, generally in a control loop, so that the volume constraint is satisfied. The most basic expression in the update is $x_i B_i^\eta$, where η is a damping factor usually taken as 0.5. The parameter ζ is a moving limit that controls the density changes from one iteration to the next and can be tuned for efficiency.

According to Ananiev [23], this update rule evokes critical questions. First, the role of the Lagrangian multipliers is not clear and contradictory. The multiplier λ is calculated at the iteration using an inner optimization loop, but the multipliers ϵ_i and κ_i that correspond to the bound constraints are ignored; if these constraints are active, the density at the bound value is fixed. It should be noted that an OC method considers only one constraint. Second, the role of the move limit ζ is unclear. It is known that it should be small but its importance has yet to be explained. Duysinx [24] makes the observation that, although there is a quick reduction in the objective values during the first 10 iterations, a slow progression toward the optimum structure follows, often requiring 50 or more iterations until convergence. Furthermore, the OC method has been known to lead to the checkerboarding phenomenon and mesh dependency problems (see Sec. II.C).

Whereas the OC method is a direct optimization scheme that calculates gradients based on KKT conditions, the HCA method for structural topology optimization is an indirect scheme. It does not require gradient information to update the material distribution at each iteration. The OC update is essentially based on the sensitivities of the elemental compliance with respect to each element x_i as expressed by $x_i B_i^\eta$. The HCA method update is based on the compliance state of each element. Various update schemes have been developed to drive the states of each material element to a target level.

One can make a comparison of the HCA and OC methods by simplifying the HCA material update rule expressed in Eq. (22) to include only the proportional control term and substituting state S in Eq. (23) with compliance. The resulting material update is

$$x_i^{(k+1)} = x_i^{(k)} + K_P(\bar{c}_i^{(k)} - c_i^{*(k)}) \tag{11}$$

where K_P is typically a value between 0.10 and 0.25 from numerical experimentation. It appears that some similarity between the HCA and OC updates exists when the first two conditions in (9) are satisfied. However, expanding the terms and replacing ζ with K_P, the update rule is

$$x_i^{(k+1)} = x_i^{(k)} \pm K_P x_i^{(k)} \tag{12}$$

From this, one can see that the OC update rule makes use of both sensitivities and densities, whereas the HCA update rule is based solely on the density states. Furthermore, filtering of the sensitivities is required to alleviate the checkerboarding phenomenon that is prevalent in all topology optimization schemes.

In the OC method presented by Bendsøe, a mesh-independence filter is used. This filter operates on sensitivity information to ensure mesh independency [17]. The heuristic filter modifies the sensitivity of each element using a weighted average of sensitivities (weights are based on relative densities) of neighboring elements. In this respect, this filtering scheme is similar to the neighborhood scheme used in the HCA method; see Eq. (21). However, the HCA method operates only on the density state of an element. The modified sensitivities resulting from the filter schemes used by Bendsøe are described as

$$\frac{\partial c'}{\partial x_i} = \frac{1}{x_i \sum_{j=1}^{N} \hat{H}_i} \sum_{j=1}^{N} \hat{H}_j x_j \frac{\partial c}{\partial x_j} \tag{13}$$

where N is the number of elements in the design domain and the convolution operator $\hat{H}_i$ is

$$\hat{H}_i = r_{\min} - \text{dist}(i,j) \qquad \{j \in N | -\text{dist}(i,j) \le r_{\min}\} \tag{14}$$

The parameter $r_{\min}$ is the filter radius, and the operator $\text{dist}(i,j)$ is defined as the distance between the centers of elements i and j. The method presented by Sigmund [14] is illustrated in Fig. 1.

IV. Method of Moving Asymptotes

Sequential convex programming (SCP) is an MP approach used for solving topology design problems. The MMA is the most popular SCP method used for structural optimization because of its efficiency. The MMA was developed by Svanberg [4]. In the MMA, a subproblem is approximated at each iteration. A strictly convex subproblem is generated based on sensitivity information at the current design and solved. The approximation has the form [17]

$$F(\boldsymbol{x}) \approx F(\boldsymbol{x}^0) + \sum_{i=1}^{n}\left(\frac{r_i}{U_i - x_i} + \frac{s_i}{x_i - L_i}\right) \tag{15}$$

where parameters r_i and s_i are determined by

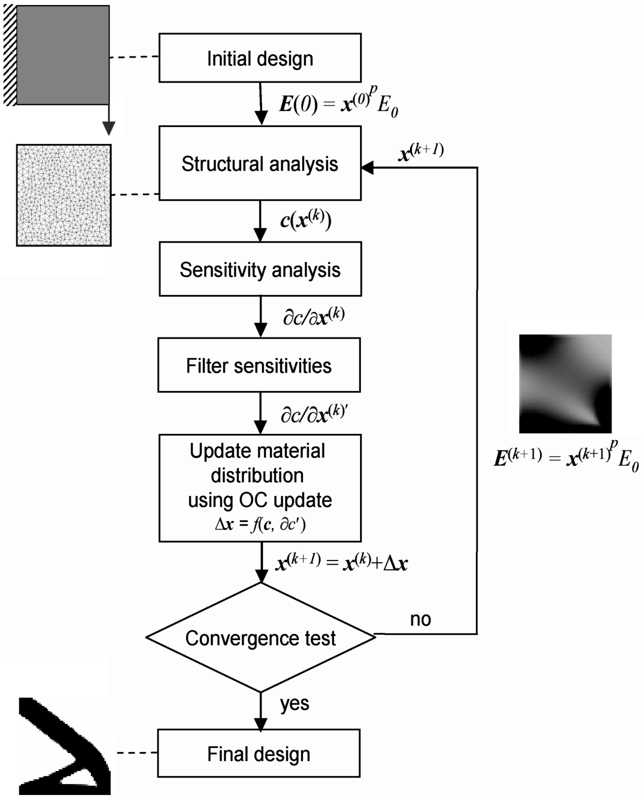


Fig. 1 Algorithm flow of the OC method for topology optimization.

$$\text{if } \left.\frac{\partial F}{\partial x_i}\right|_{x=x^0} \geq 0 \text{ then } r_i = (U_i - x_i^0)^2 \left.\frac{\partial F}{\partial x_i}\right|_{x=x^0} \text{ and } s_i = 0$$
$$\text{if } \left.\frac{\partial F}{\partial x_i}\right|_{x=x^0} < 0 \text{ then } r_i = 0 \text{ and } s_i = (x_i^0 - L_i)^2 \left.\frac{\partial F}{\partial x_i}\right|_{x=x^0} \tag{16}$$

The generation of these subproblems is controlled by the "moving asymptotes" U_i and L_i. This is similar to the move limits in sequential linear programming. These parameters control the range for which the approximation of the function $F(\boldsymbol{x})$, shown in Eq. (15), is reasonably accurate. The main characteristics of the MMA are that the necessary conditions for the optimality of each subproblem do not couple the design variables and that the convex approximations allow for the use of primal-dual methods. These two aspects greatly reduce the computational time required compared with other gradient-based algorithms. The MMA subproblem for the topology optimization problem at the kth iteration has the form

$$\min_{x_i} \; c(x_i^{(k)}) - \sum_{i=1}^{N} \frac{(x_i^{(k)} - L_i)^2}{x_i - L_i} \left.\frac{\partial c}{\partial x_i}\right|_{x=x^0}$$
$$\text{s.t.} \; \sum_{i=1}^{N} x_i v_i \leq V$$
$$0 < x_i \leq 1 \tag{17}$$

where the design variable x_i is defined by Eq. (2) and N is the number of material elements in the design domain. The Lagrangian function can be written as

$$\mathcal{L} = c(x_i^{(k)}) - \sum_{i=1}^{N} \frac{(x_i^{(k)} - L_i)^2}{x_i - L_i} \left.\frac{\partial c}{\partial x_i}\right|_{x=x^0} + \lambda \left(\sum_{i=1}^{N} x_i v_i - V \right) \tag{18}$$

According to Bendsøe and Sigmund [17], for the case $L_i = 0$, move limit $\zeta = \infty$, and tuning parameter $\eta = 0.5$, the MMA update is equivalent to the OC update given in Eq. (9).

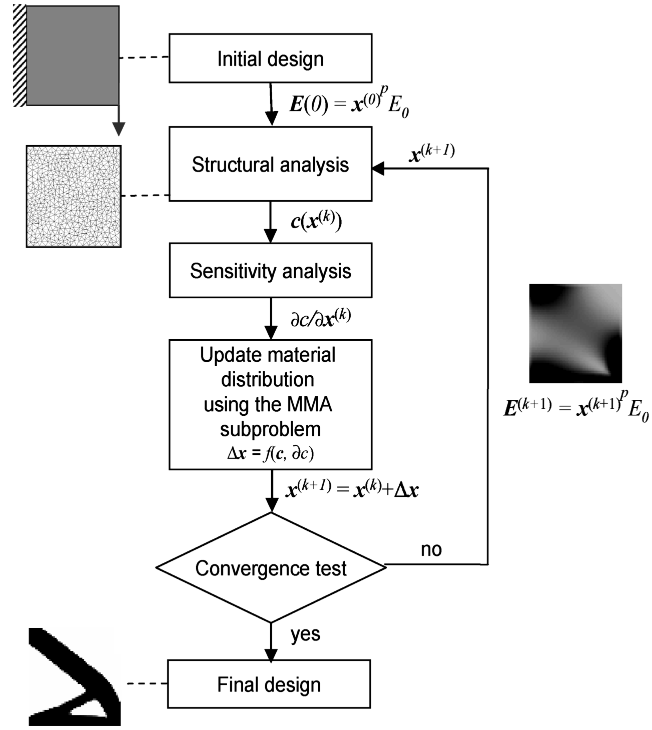


Fig. 2 Algorithm flow of the MMA for topology optimization.

The OC method and the MMA are computationally similar for minimum compliance problems, which can be illustrated by solving the MMA problem using a dual method [17]. Dual methods allow for optimization problems with a large primal formulation to be stated in a dual space that has a reduced number of design variables. The dual space is limited to the number of active constraints and, consequently, is small in the application to topology optimization [24]. For simple minimum compliance problems, the MMA takes longer to converge to a solution but can handle any number of constraints, whereas the OC method is only used for single constraint problems [17]. The MMA is illustrated in Fig. 2.

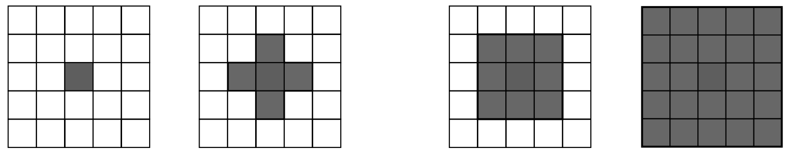
V. Topology Synthesis Using Hybrid Cellular Automata

The HCA method is a computational technique that can be used to synthesize optimal topologies without the need for sensitivity information. Although the HCA method is not explicitly an optimization technique, the local rules produce this behavior. This approach is inspired by the biological process of bone remodeling as presented by Tovar et al. [12]. First, we must understand the concept of CA and how it is used to model physical phenomena.

A. Cellular Automata

CA is a discrete model studied in mathematics that consists of an infinite, regular grid of cells, or lattice, in which each is characterized by a finite number of states [25]. The state of each cell at a given time, or generation, is a function of the states of a finite number of neighboring cells, called the neighborhood. Every cell has the same set of rules, which are applied based on the information in its neighborhood. These rules are applied to the entire CA lattice for each generation.

The notion of CA was initially conceived by John von Neumann in the late 1940s. According to Burks [26], the first CA proposed by von Neumann was a two-dimensional square lattice composed of several thousand cells. The CA rule made use of the state of each (central) cell plus the states of its four nearest neighbors located directly to the north, south, east, and west. Each cell had from 3 to 29 possible states. This CA model was so complex that it has only been partially implemented on a computer. The von Neumann rule has the so-called



a) Empty ($\hat{N}$=0) b) Von Neumann ($\hat{N}$=4) c) Moore ($\hat{N}$=8) d) Extended ($\hat{N}$= 24)

Fig. 3 Typical 2-D neighborhoods in CA. $\hat{N}$ is the number of neighboring cells.

property of universal computation, meaning that there exists an initial configuration of the CA that leads to the solution of any computer algorithm. Accordingly, any universal computer circuit (i.e., logical gate) can be simulated by the rule of the automaton. This illustrates that complex and unexpected behavior can emerge from a CA rule. Cellular automata have also been used to study artificial life to better understand real life and the behavior of living species through computer models. The most famous model using CAs is John Conway's "Game of Life," [27] used to model population behavior using five simple rules.

The first applications of a pure CA approach to structural design were presented by Abdalla and Gürdal [28], Missoum et al. [29], and Inou et al. [30,31]. In these approaches, the structural analysis is performed using CA models and generally requires tens of thousands of iterations to converge to a solution. CAs have been applied to both discrete and continuous structures. Gürdal and Tatting [32] and Slotta et al. [33] applied CA to truss structures. In those applications, a rectangular design domain was composed of an array of truss elements. Each cell was composed of a node and the eight trusses from neighboring nodes in an 45-deg arrangement. Kita and Toyoda [34] presented a similar methodology to the HCA method developed by Tovar in that it used the FEM for structural analysis. The local update rule is based on the minimization of both the weight of the structure and the deviation between the yield stress and the von Mises equivalent stress for each cell. Furthermore, a two-dimensional isotropic material is considered for which the thickness of each CA cell is the design variable. Hajela and Kim [35] used a genetic algorithm based on energy minimization to determine an appropriate rule for a two-dimensional continuum.

B. Hybrid Cellular Automaton Method

The HCA method differs from the CA approaches in that it uses the FEM for structural analysis; hence, it is a *hybrid* approach because each cell is provided global information. As in the previous methods presented, the design domain is discretized in material elements, and continuum finite elements (FE) are used model the structure. The states of the material elements in the design domain are represented using a lattice of cells, in which a one-to-one correspondence between the cells and the FEs in the finite element model generally exist, although this is not required. However, uniformity in the CA lattice is required.

In the HCA method, the state of each cell is defined by design variables x_i (e.g., density) and field variables S_i (e.g., compliance). The complete state of each cell at a time/iteration k is expressed by

$$\boldsymbol{\beta}_i^{(k)} = \begin{Bmatrix} S_i^{(k)} \\ x_i^{(k)} \end{Bmatrix} \tag{19}$$

For the linear-static problem, the method synthesizes a topology by using a CA model that mimics the following problem:

$$\begin{aligned} \min_{x_i} \quad & \sum_{i=1}^{N} |S_i(x_i) - S_i^*| \\ \text{s.t.} \quad & \sum_{i=1}^{N} x_i v_i = V \\ & \mathbf{Kd} = \mathbf{F} \\ & 0 < x_i \leq 1 \end{aligned} \tag{20}$$

where S_i is the field variable state and S_i^* is the field state target (or set point) of a cell. A set of local rules is used to determine the material distribution at each iteration. These rules are applied to the local information collected in the neighborhood of each cell in the CA lattice.

1. Neighborhoods

The *effective* field state of a cell is defined by the state of itself and the states of the cells within the neighborhood. This can be written as

$$\bar{S}_i^{(k)} = \frac{S_i^{(k)} + \sum_{n \in N(i)} S_n^{(k)}}{\hat{N} + 1} \tag{21}$$

where $S_n^{(k)}$ is state of the nth neighbor at the kth iteration, $N(i) = \{\text{indices of neighbors of the } i\text{th cell}\}$, and $\hat{N}$ is the total number of neighbors in the neighborhood. This use of a neighborhood can be viewed as a filtering technique that prevents checkerboarding and mesh dependency. In the CA paradigm, the same neighborhood is applied to all of the cells in the lattice. In the context of structural optimization, no state information exists outside of the design domain. Therefore, the neighborhood is modified for the boundary elements to include only neighbors within the design region. The neighborhood scheme represents a local filter on the state information of each material element. Figure 3 depicts some common two-dimensional neighborhood layouts.

2. Material Distribution Rules

In CA, the rules operate on the set of the states of the cells and govern the global behavior of the CA lattice. The rules control material distribution. From a fully stressed design [36], the HCA method allocates material based on the compliance level of each element. An inversely proportional relationship exists between the elastic modulus and compliance, that is, for an elastic body under loading, as the modulus decreases, the compliance increases (for the same loading). Therefore, [using the power law relationship in Eq. (3)], relative density must be added to reduce the compliance of an element; to increase compliance, density is removed. The set point directly controls the total mass distributed within the design domain, as there is a one-to-one correspondence between the compliance of the structure and the total mass of the structure when no mass control is employed.

Although numerous distribution rules can be used [12], a control-based rule that uses proportional, integral, and derivative feedback is used in this paper to increase the efficiency in reaching a solution. Using the material update, the change in relative density x_i of element i at the kth iteration can be expressed as

$$\Delta x_i^{(k)} = \max\left\{-0.2, \min\left\{K_P e_i^{(k)} + K_I \int_0^k e_i^{(\kappa)}\, \mathrm{d}\kappa + K_D \Delta e_i^{(k)}, 0.2\right\}\right\} \tag{22}$$

where e_i is the error between the cell field state, and K_P, K_I, and K_D are the proportional, integral, and derivative control gains, respectively. The control-based rule is normalized by the set point. The maximum allowable change in relative density of any element is 0.2. The gain parameters scale each term in the update rule and can be determined by numerical experimentation. Furthermore, in this

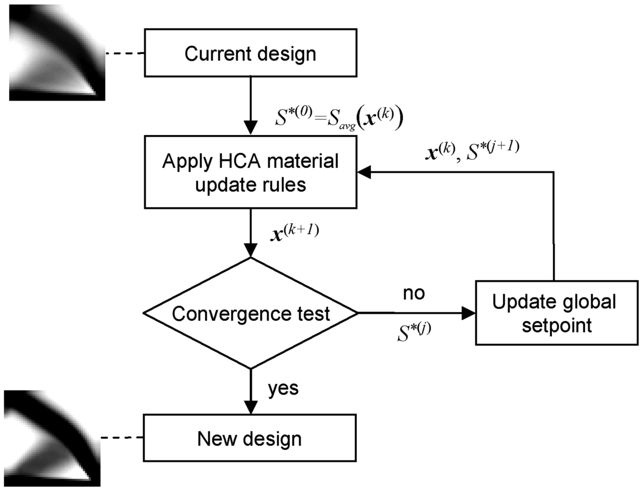


Fig. 4 Illustration of the HCA material update for mass control using a set point update strategy.

paper, the integral error is limited to the error from the previous iteration. The error in the update rule is calculated as

$$e_i^{(k)} = \bar{S}_i^{(k)} - S_i^{*(k)} \tag{23}$$

Furthermore, the field state S_i is represented by the compliance c of the ith element,

$$S_i = c_i = x_i^p \mathbf{u}_i^T \mathbf{K}_i \mathbf{u}_i \tag{24}$$

where $\mathbf{u}_i$ is the displacement vector for the nodes of the ith FE and $\mathbf{K}_i$ is the elemental stiffness matrix. Accordingly, the set point S^* can be written as c^*.

3. Mass Control

In this paper, a mass control scheme is applied to the HCA method for comparison with the OC method and the MMA. For a given design domain and set of boundary conditions, a decreasing monotonic relationship exists between the final mass and the compliance of the structure [37]. The higher the set point, the lower the mass and vice versa. At a given iteration, the difference between the current field state of a cell and the set point directly affects the

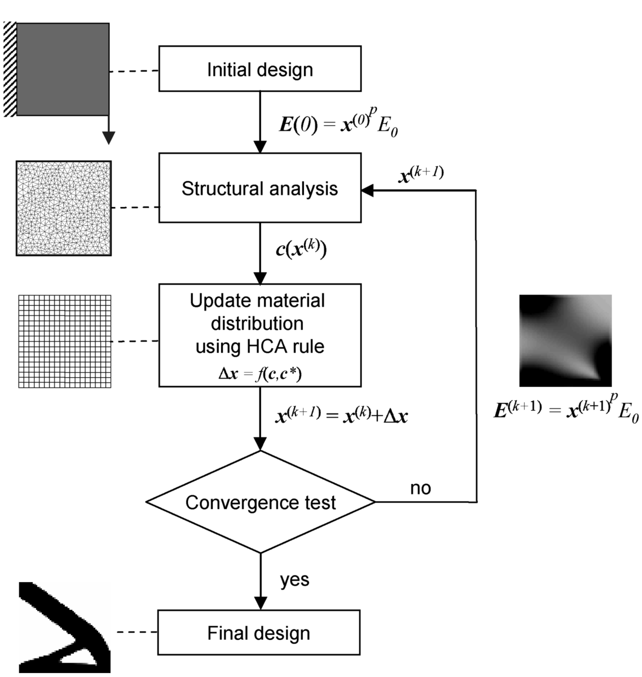


Fig. 5 Illustration of the HCA method for topology synthesis.

Table 1 Material parameters for the example problems

Parameter	Value
Elastic modulus (E_0)	1
Poisson's ratio (ν)	0.3
Penalization (p)	3

material distribution within the design domain. Therefore, to design for a specific mass, the corresponding set point must be determined.

A scheme for finding this set point can be accomplished by simply iterating on the HCA update rules and updating the set point, as shown in Fig. 4, until the correct mass results after applying the design rule expressed in Eq. (22). The set point for the $k+1$ HCA iteration is found by iterating on the update

$$S^{*(j+1)} = S^{*(j)}\left(M_f^{(k+1)}/M_f^*\right) \tag{25}$$

where j is an iterator for the subloop on the HCA rules in Eq. (22) and M_f^* is the mass fraction target. The mass fraction of a design domain is defined as

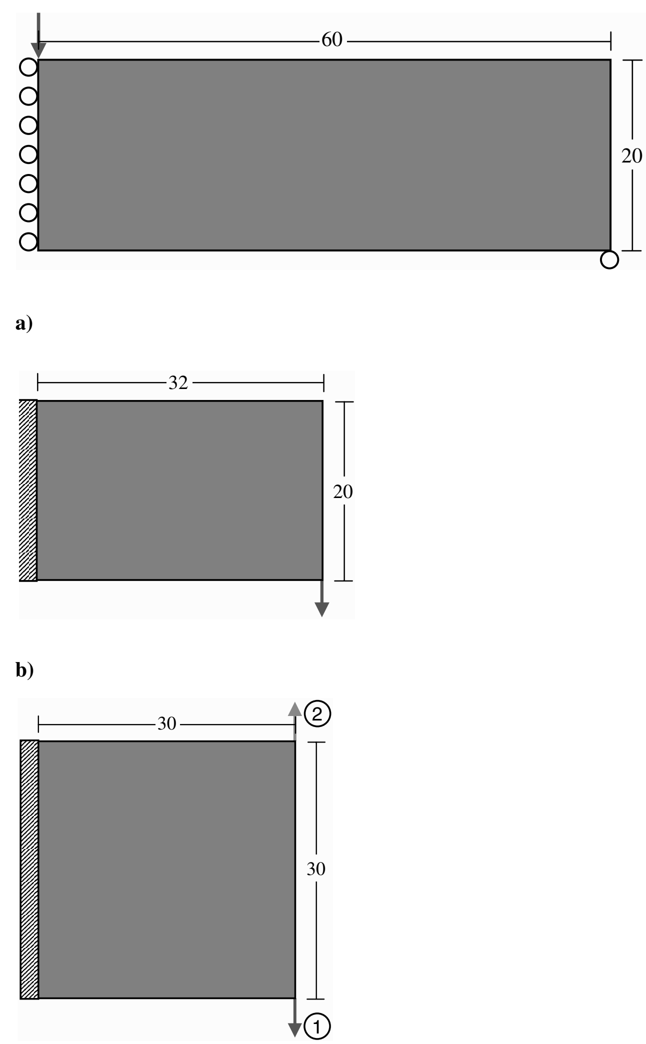


Fig. 6 Sample design domains: a) MBB beam using a single load (half-span modeled due to symmetry), b) a cantilever beam, and c) a cantilever beam modeled for two load cases (each load has equal weight). Unit loads are applied.

Table 2 Topology comparison for the HCA and OC methods and the MMA using global convergence (the optimization performance is tabulated in Table 3)

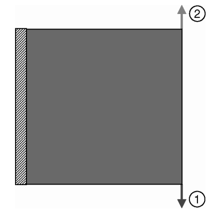
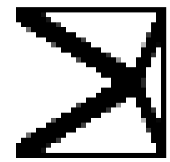
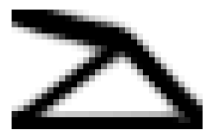
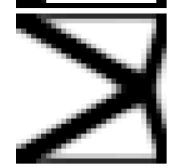
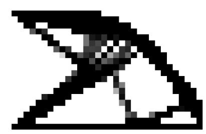
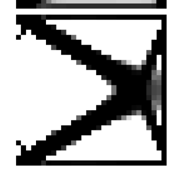
Method			
HCA			
OC			
MMA			

Table 3 Topology comparison study results. The design domain discretization for sample problem a is 60 × 20, sample problem b is 32 × 20, and sample problem c is 40 × 40. Global convergence parameters: design domain a $\varepsilon = 0.64$, b $\varepsilon = 0.9$, and c $\varepsilon = 1.2$

Design domain	Method	No. iterations	Compliance	No. interim den.
	HCA	38	191.7	42
a	OC	54	203.9	444
	MMA	83	203.1	120
	HCA	24	54.1	31
b	OC	57	59.5	259
	MMA	67	60.4	54
	HCA	22	57.0	32
c	OC	50	60.4	106
	MMA	36	65.9	402

$$M_f = \frac{\sum_{i=1}^{N} x_i}{N} \tag{26}$$

When the mass of the structure at the kth iteration satisfies the mass target, the resulting material update control loop is terminated. Thus, the mass constraint is enforced at each HCA iteration.

4. *Methodology*

In the HCA method, the density state of each CA cell in the design domain is modified to drive the compliance of the element to the compliance set point. The local control-based rules are used to appropriately add mass to elements above the specified level of compliance and decrease mass in elements that are below this set point. Adopting the idea of a fully stressed design, all elements in the structure should contribute to the potential energy stored in the structure for a given loading. Therefore, a single target set point is applied to all of the cells in the design domain. The HCA method for topology synthesis, illustrated in Fig. 5, is described as follows:

Step 1) Define the design domain, material properties, load conditions, and initial design, $\boldsymbol{x}^0$.

Step 2) Evaluate the compliance, $c_i^{(k)}$, at each discrete location, i, using the FEM.

Step 3) Update the material distribution according to Eq. (23) to obtain the new design, $x_i^{(k+1)}$.

Step 4) Check for convergence. If the convergence criterion is satisfied, the final topology is obtained; otherwise, the iterative process continues from Step 2.

VI. Numerical Examples

In this paper, three examples are used to demonstrate the difference between the performances of the HCA and the OC methods and the MMA. The first problem is a Messerschmitt–Bölkow–Blohm (MBB) beam using a single load (half-span modeled due to symmetry) that is subject to an equality mass fraction constraint, $M_f^* = 0.5$. The second problem is a cantilever beam, and the third problem is a cantilever beam modeled for two load cases. The second and third problems are subject to a mass constraint of 0.4. Unless otherwise stated, the topologies generated using the HCA method with the von Neumann neighborhood ($\hat{N} = 4$) to maintain consistency in the comparisons and gain parameters of $K_P = 0.25$, $K_I = 0.10$, and $K_D = 0.05$ to define the mass update. In the examples presented, the filtering parameter $r_{\min} = 1.5$ is used unless otherwise stated.

To demonstrate the differences between the three methods presented in this paper, the performance of each is studied. Four-node FEs are used in the linear-static FEA. Unless otherwise specified, the parameters listed in Table 1 are used. The basic performance of each method is studied in terms of 1) the number of iterations required to reach a solution, 2) the final compliance of the solution (without postprocessing), and 3) the number of intermediate elements in the final topology. The stopping criterion used for the HCA method is global in nature in that the synthesis process terminates when the total mass change in the design domain is within some tolerance:

$$\sum_{i=1}^{N} (|x_i^{(k)} - x_i^{(k-1)}|) < \varepsilon \quad \text{where } \varepsilon = 0.001 \times N_X \times N_Y \tag{27}$$

where N_X and N_Y are the number of elements along the x and y dimensions. The OC method and the MMA traditionally use a local stopping criterion:

Table 4 Topology comparison for the HCA and OC methods and the MMA using local convergence (the optimization performance is tabulated in Table 5)

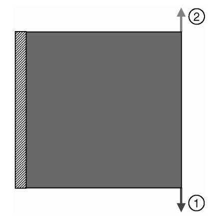
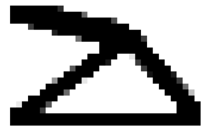
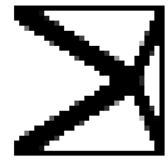
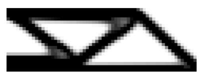
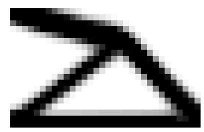
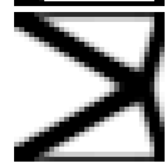
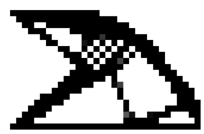
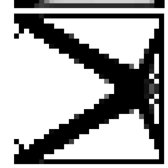
Method			
			
HCA			
OC			
MMA			

Table 5 Tabulated topology comparison study results using local convergence

Method	Design domain	No. iterations	Compliance	No. interim den.
	a	141	191.3	38
HCA	b	80	54.1	24
	c	40	57.0	30
	a	94	203.3	432
OC	b	61	59.5	260
	c	41	65.9	402
	a	125	201.8	73
MMA	b	117	59.7	8
	c	83	59.4	46

$$\max\{|x_i^{(k)} - x_i^{(k-1)}|\} < \varepsilon \quad \text{where } \varepsilon = 0.001 \tag{28}$$

Both stopping criteria are investigated. In this paper, the topologies generated using the HCA and the OC methods and the MMA for the three sample problems shown in Fig. 6 are presented. The effects of penalization on the topology synthesis and performance along with the dependence of mesh discretization on the final topology are studied as well.

A. Basic Comparison

Tables 2 and 3 present the results using the global stopping criterion. Table 3 tabulates the three measures of performance considered in this study. Using the local stopping criterion, we obtain some refinement of the topologies, shown in Table 4. The quantitative results for the topologies generated are tabulated in Table 5. We see that the structures generated using the HCA method with the local convergence criterion have a lower compliance. The topologies generated using MMA generally have more stiffness than those generated using the OC method, although this is a result of the checkerboards in the topology, which results from the lack of filtering.

The OC method converges to a solution the quickest using the local stopping criterion; however, the HCA method shows the

Table 7 Filtering study for the OC method (the global convergence criterion is used)

Method	Filter value	No. iterations	Compliance	No. interim den.
	None	26	204.0	146
OC	1.2	68	189.7	1754
	1.5	96	190.8	2168
	2.0	79	193.0	2515

Table 8 Results of filtering study for MMA (the global convergence criterion is used)

Method	Filter value	No. iterations	Compliance	No. interim den.
	None	89	202.5	658
MMA	1.2	>1000	189.5	1666
	1.5	>1000	191.4	2083
	2.0	>1000	193.3	2394

Table 6 Filtering study for the HCA method (the global convergence criterion is used)

Method	Neighborhood	No. iterations	Compliance	No. interim den.
	None	13	213.2	21
HCA	von Neumann	44	193.6	200
	Moore	36	192.7	189
	Extended Moore	30	194.3	293

a) b) c) d)

Fig. 7 Generated HCA topologies for design domain b using differing neighborhoods: a) empty ($\hat{N} = 0$), b) von Neumann ($\hat{N} = 4$), c) Moore ($\hat{N} = 8$), and d) extended Moore ($\hat{N} = 24$). A design domain discretization of 180 × 60 is considered.

a) b) c) d)

Fig. 8 Generated OC topologies for design domain b using differing filter values ($r_{\min}$): a) $r_{\min} = 1.0$ (no filter), b) $r_{\min} = 1.2$, c) $r_{\min} = 1.5$, and d) $r_{\min} = 2.0$.

a) b) c) d)

Fig. 9 Generated MMA topologies for design domain b using differing filter values ($r_{\min}$): a) $r_{\min} = 1.0$ (no filter), b) $r_{\min} = 1.2$, c) $r_{\min} = 1.5$, and d) $r_{\min} = 2.0$.

greatest computational efficiency in achieving a final topology using the global criterion. For these coarse design domain discretizations, there is little to no change between the topologies obtained using the global and local convergence criteria. This is further supported by the quantitative results in Tables 3 and 5. Using the HCA method, the largest reduction in compliance is 0.4 with a reduction of 4 intermediate densities (see sample problem b, although this small reduction is seen after 106 more iterations).

B. Influence of Filtering

All topology optimization techniques typically exhibit mesh dependence and checkerboard patterns. To reduce this, filtering techniques are required. However, filtering can lead to suboptimal solutions. The HCA method filters on the field state (i.e., compliance) of each element. This local filtering is performed using neighborhoods as described in Sec. V.B.1. The decision about material placement in the HCA method is based on the effective field state of the element, which is the average of the field state of the element and its neighbors. The influence of the neighborhood for design domain b is illustrated in Fig. 7. The convergence time and a comparison of the compliance of the resulting topologies are compiled in Table 6.

As expected, checkerboarding occurs when no filtering ($\hat{N} = 0$) is applied. The final topology is relatively insensitive to the neighborhood configuration used. The general trend that one observes is that the topology becomes "thicker," or more qualitatively robust, as the neighborhood size increases. However, overfiltering can be an issue for smaller design domains, and the appropriate neighborhood size should be used relative to the design domain size.

Gradient-based schemes use sensitivity information to determine the material distribution. Therefore, the filter is typically performed on the sensitivity information of each element, such as in the OC method. To illustrate the need for filtering using the OC method, the filtering control parameter $r_{\min}$ was varied. The resulting topologies are as shown in Fig. 8. Again, checkerboarding results when no filtering is applied. From Table 7, it is observed that, as the filtering parameter increases, the number of intermediate densities increases. As stated before, overfiltering can be an issue.

The MMA used in this paper does not include a filter scheme in the material update, as evidenced by the checkerboarded topologies shown in previous results. The topologies in Fig. 9 are generated

Table 9 Penalization study results using global convergence. The design domain discretization for sample problem a is 128 × 80 (the optimization performance is tabulated in Table 10)

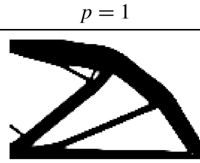
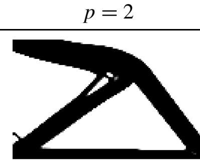
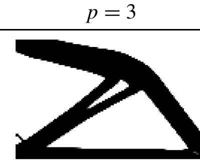
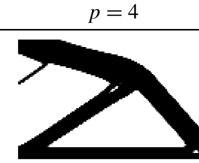
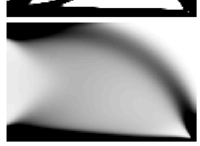
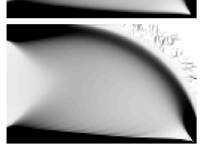
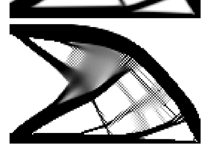
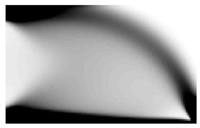
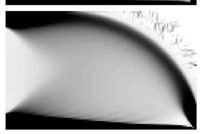
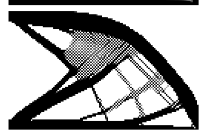
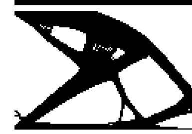
Method	$p = 1$	$p = 2$	$p = 3$	$p = 4$
HCA				
OC				
MMA				

Table 10 Penalization study results using global convergence ($\varepsilon = 10.24$)

Method	Penalization (p)	No. iterations	Compliance	No. interim den.
	1	18	54.0	246
HCA	2	29	55.6	197
	3	39	55.5	172
	4	43	55.4	170
	1	23	48.3	8673
OC	2	62	55.0	2228
	3	40	56.2	1775
	4	38	56.4	1307
	1	>1000	48.4	8477
MMA	2	151	56.1	1721
	3	70	61.0	879
	4	51	65.8	421

Table 11 Penalization study results using local convergence. The design domain discretization for sample problem a is 128 × 80 (the optimization performance is tabulated in Table 12)

Method	$p = 1$	$p = 2$	$p = 3$	$p = 4$
HCA				
OC				
MMA				

using the MMA with the linear filter expressed in Eqs. (13) and (14). Filtering on sensitivities does not work well because it causes the method to add material along the boundary of the topology, regardless of whether there is material or not. This phenomenon occurs regardless of the filtering radius; see Table 8. The cost of filtering sensitivities in the MMA is the nonconvergence of the method. Using both the OC method and the MMA, the compliance of the final structure without filtering is higher compared with when filtering is used due to the large number of intermediate densities resulting in a compliant structure with low stiffness. Even though the selection of the filtering radius depends on the problem, in every case, as $r_{\min}$ increases, the number of intermediate densities increases as well. As a result, the compliance of the final structure increases. This demonstrates overfiltering. In the HCA method, this trend is not observed for this particular problem. As in the neighborhood filtering in the HCA method, the OC method and the MMA topologies are relatively insensitive to $r_{\min}$.

C. Influence of the Power Law

Using the density approach, specifically the SIMP power law, for material parameterization, it is usually necessary to use a power

Table 12 Penalization study results using local convergence

Method	Penalization (p)	No. iterations	Compliance	No. interim den.
	1	725	53.6	143
HCA	2	>1000	54.4	132
	3	>1000	54.8	121
	4	>1000	54.8	110
	1	26	48.3	8647
OC	2	428	54.7	2321
	3	267	55.4	1853
	4	317	56.2	1488
	1	>1000	48.4	8477
MMA	2	356	54.3	486
	3	207	60.2	14
	4	134	65.0	3

Table 13 Influence of the HCA neighborhood on the final topology for the increment of power p. The design domain discretization for sample problem a is 128 × 80 (the optimization performance is tabulated in Table 12)

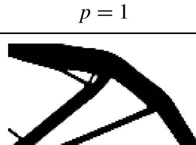
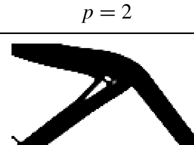
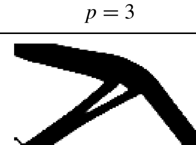
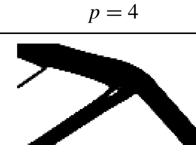
$\hat{N}$	$p = 1$	$p = 2$	$p = 3$	$p = 4$
4				
8				
24				

Table 14 Penalization-neighborhood study results using global convergence

$\hat{N}$	Penalization (p)	No. iterations	Compliance	No. interim den.
	1	18	54.0	246
4	2	29	55.6	197
	3	39	55.5	172
	4	43	55.4	170
	1	18	54.0	241
8	2	25	55.3	171
	3	36	55.2	147
	4	36	55.6	154
	1	19	54.3	323
24	2	22	55.6	286
	3	25	55.3	200
	4	33	55.7	186

Table 16 Mesh dependency study results using global convergence

Method	Discretization	No. iterations	No. interim. den.	Eval. time, s[a]
	30 × 30	22	32	14.0
HCA	60 × 60	25	74	152.4
	120 × 120	16	190	1339.0
	240 × 240	16	530	19,989.7
	30 × 30	36	402	26.8
OC	60 × 60	35	880	253.1
	120 × 120	42	1870	8460.9
	240 × 240	44	5058	48,534.0
MMA	30 × 30	50	106	34.1
	60 × 60	49	218	325.3
	120 × 120	50	796	3636.9
	240 × 240	51	2658	54,803.0

[a]Intel Pentium 4, 3.0 GHz processor

greater than 1 to drive the design variables to their boundary so as to arrive at a 0–1 solution, see Tables 9 and 11. The corresponding quantitative results are tabulated in Tables 10 and 12. Using the global stopping criterion originally developed for the HCA method, one obtains the topologies shown in Table 9. Using the local stopping criterion, one obtains slightly different topologies; see Table 11. From Tables 9 and 11, we see that only the HCA method synthesizes a viable solution using $p = 1$, whereas the OC method and the MMA require $p > 1$ to generate a topology. The HCA method is able to generate 0–1 topologies because the compliance state of every element in the design domain is driven toward an optimum state by modifying the material property (elastic modulus as a function of the relative density) regardless of penalization. The majority of the elements saturate, that is, achieve the maximum or minimum material property value, before reaching their optimum states. Therefore, the relative densities of these elements are driven to 0 or 1. A few intermediate elements at the interface adjacent to the 0 and 1 elements are able to achieve their optimum state before saturation.

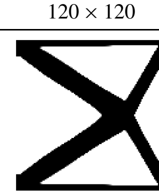
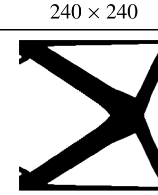
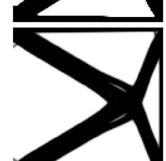
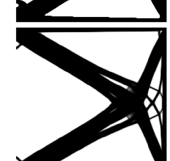
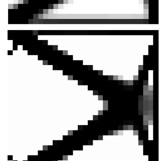
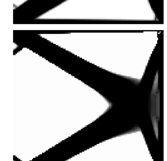
Although 0–1 topologies can be synthesized with the HCA method using $p = 1$, they are more efficiently generated in terms of the number of iterations needed to achieve a solution at a higher penalization power. However, higher penalization is more likely to result in a suboptimal topology. Using higher values of p with the OC method and the MMA results in a better convergence time and fewer intermediate densities. However, the HCA method is more efficient in terms of the number of iterations required to reach a final a topology using the global convergence criterion. This is not necessarily the case using the local convergence criterion as shown in Table 12. Note that the topologies generated by the OC method in Table 9 contain a large number of intermediate densities, although this is not necessarily apparent by inspection.

Table 13 shows that the influence of the penalization power p is not sensitive to the range of the filtering that could be applied using the HCA method. The quantitative results are shown in Table 14. The relative insensitivity to p is a great advantage in multiphysics problems in which the relationship between the design variable and material properties is not known. Comparing the 32 × 20 structure in Table 4 with the higher resolution 128 × 80 structure in Table 13, one observes that obtaining information from a larger neighborhood reduces the mesh dependency. Again, the choice of neighborhood size is dependent on the discretization of the design domain.

D. Mesh Dependency

The dependence of the mesh discretization on the final topology using design domain c is shown in Table 15. The OC method used in this paper uses the mesh-independence scheme expressed in Eq. (13) to alleviate this phenomenon. In this study, only the global convergence criterion is considered. Starting from a coarse discretization of the design domain, Table 15 shows that, although all three schemes produce increasingly detailed structures as the discretization increases, from a qualitative perspective, the HCA method results in the same basic topology even for the largest discretization. This mesh independency may result from using information from neighboring elements. The results are shown in Table 16.

Table 15 Mesh dependency study results using global convergence (a power law penalization $p = 3$ is used)

Method	30 × 30	60 × 60	120 × 120	240 × 240
HCA				
OC				
MMA				

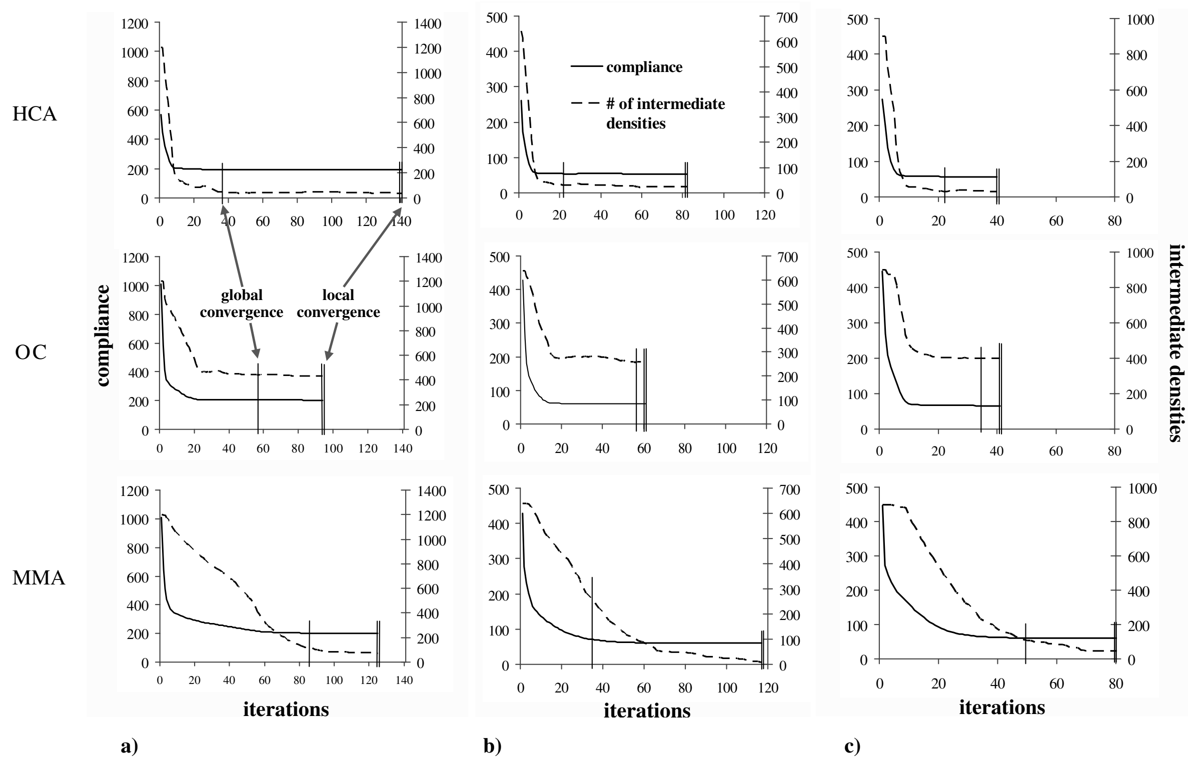


Fig. 10 Comparison of the iteration histories for the sample problems a, b, and c in Sec. VI.A. The compliance of the topology and the total number of intermediate densities at each iteration are plotted.

VII. Conclusions

This paper presents a comparison between the HCA and OC methods and the MMA. The HCA and OC methods produce similar results for the simple problems presented. Using the global convergence criterion, the HCA approach generated a solution faster than the OC method or the MMA for the problems presented. In general, accounting for the skewedness of the mean compliance values due to intermediate densities, the topologies generated are similar in compliance under the prescribed loading. However, the HCA method results in fewer intermediate density elements compared with the topologies generated using the MMA and the OC approach. The final topologies produced by the OC approach contain the highest number of intermediate density elements, which are primarily located along the boundaries of the topologies. This is due to the filtering scheme applied to the sensitivities. The lack of filtering in the MMA produces fewer intermediate densities compared with the OC method, although checkerboarding results.

Table 3 shows that the HCA method converges to a solution in the fewest number of iterations for the example problems presented using the global convergence criterion.. Even though a convergence criterion based on local information is most appropriate for the HCA method, Fig. 10 illustrates that little is gained in terms of the objective function and the number of intermediate densities for the HCA method. There is little reduction in the compliance of the structure and in the intermediate densities for the HCA and OC methods, although this is not the case with the MMA. The benefits of using global convergence criterion are seen in large problems, such as the 240×240 mesh discretization for sample problem c, shown in Table 16. The efficiency demonstrated by the HCA method in terms of the number of required iterations is a result of the choice of gains used in the material update in Eq. (22). The best set of gains is problem dependent, although the gains used in this paper work well for general problems. Furthermore, this study demonstrates that 1) the HCA method can be used with a linear material parametrization whereas the OC method and the MMA require a higher-order parametrization, and 2) the HCA method and the MMA generate approximately the same basic topology, independent of the design domain discretization. However, the MMA produced a large number of checkerboards in the topologies. Although the MMA does not need a mesh dependency filter, it would benefit from a filter to reduce checkerboarding.

Finally, because the HCA method is a nongradient method, it can be applied directly to more complex topology optimization problems, such as compliant mechanism synthesis using nonlinear analysis and crashworthiness design using dynamic analysis. In these types of problems, the sensitivities needed by the gradient-based methods are difficult to obtain without simplification of the sensitivity derivation.

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